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A Fixed Point Theorem for Weakly C-Contraction Mappings of Integral Type

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ABSTRACT: In the present paper, we shall prove a fixed point theorem by using generalized weak C-contraction of integral type. Our result is generalization of much known results.

Key words: Metric space, fixed point, weak C- contraction.

I. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a complete metric space and T: X→ X a self-map of X. Suppose that $F_f = \{x \in X | T(x) = x\}$ is the set of fixed points of f. The classical Banach's fixed point theorem is one of the pivotal results of functional analysis.

By using the following contractive definition: there exists k$\in$ [0, 1) such that $\forall$ x, y $\in X$, we have

$$d(Tx,Ty) \leq kd(x,y) \quad (1.1)$$

If the metric space (X,d) is complete then the mapping satisfying (1.1) has a unique fixed point . Inequality (1.1) implies continuity of T. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity .

Kannan [10,11] established the following result in which the above question has been answered in the affirmative.

If $T : X \to X$ where (X,d) is complete metric space , satisfies the inequality

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)] \quad (1.2)$$

where $0< k < \frac{1}{2}$ and x, y $\in X$, then T has a unique fixed point.

The mapping T need not be continuous .The mapping satisfying (1.2) are called Kannan[10,11] type mappings. There is a large literature dealing with Kannan type mappings and their generalization some of which are noted in [8] ,[17] and [19].

A similar contractive condition has been introduced by Chatterjee [6].. We call this contraction a C- contraction.

Definition 1.1.1. C-contraction

Let $T: X \to X$ where (X, d) is a metric space is called a C – contraction if there exists $0< k <\frac{1}{2}$ such that for all x, y $\in X$ the following inequality holds:

$$d(Tx,Ty) \leq k[d(x,Ty) + d(y,Tx)] \quad (1.3)$$

Theorem 1.1.1. A C- contraction defined on a complete metric space has a unique fixed point.

In establishing theorem 1.1.1 there is no requirement of continuity of the C-contraction.

It has been established in [15] that inequalities (1.1),(1.2) and (1.3) are independent of one another. C- Contraction and its generalizations have been discussed in a number of works some of which are noted in [4] ,[8], [9] and [19].

Banach's contraction mapping theorem has been generalized in a number of recent papers. As for example, asymptotic contraction has been introduced by Kirk [12] and generalized Banach contraction conjecture has been proved in [1] and [14].

Particularly a weaker contraction has been introduced in Hilbert spaces in [2]. The following is the corresponding definition in metric space.

Definition 1.1.2. Weakly contractive mapping

A mapping $T : X \to X$ where (X,d) is complete metric space is said to be weakly contractive if $d(Tx,Ty) \leq d(x,y) - \Psi\big(d(x,y)\big)$, (1.4)

Where x, y $\in X$, Ψ :[0,∞) → [0,∞) is continuous and non-decreasing,

ψ (x) = 0 if and only if x = 0 and $\lim_{x\to\infty}\psi(x) = \infty$.

There are a number of works in which weakly contractive mappings have been considered. Some of these works are noted in [3],[7],[13], and [16].

In the present work in the same spirit we introduce a generalization of C- contraction.

Definition 1.1.3. Weak C- contraction:

A mapping $T: X \to X$, where (X, d) is a metric space is said to be weakly C – contractive or a weak C-contraction if for all x, y $\in X$,

$$d(Tx,Ty) \leq \frac{1}{2}\,[d(x,Ty)+d(y,Tx)] - \Psi(d(x,Ty), d(y,Tx)) \qquad (1.5)$$

where $\Psi: [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that Ψ (x, y) = 0 if and only if x = y= 0 .

If we take Ψ (x, y) = k(x+y) where $0< k < \frac{1}{2}$ then (1.5) reduces to (1.4), that is weak C – contractions are generalizations of C – contractions.

In a recent paper of Branciari [20] obtained a fixed point result for a single mapping satisfying an analogue of a Banach's contraction principle for integral type inequality as below: there exists c $\in$ [0,1) such that $\forall$ x, y $\in X$, we have

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq k\int_0^{d(x,y)} \varphi(t)dt$$

Where $\varphi : R^+ \to R^+$ is a Lebesgue – integrable mapping which is summable, non-negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$.

Our main result is extended and modified to the weak C – contraction mapping in integral type.

II. MAIN RESULT

Theorem 2. Let $T: X \to X$ where (X,d) is complete metric space be a weak C-contraction, which is satisfying the following property:

$$\begin{aligned}\int_0^{c(Tx,Ty)} \varphi(t)dt \leq\ & \alpha \int_0^{d(x,Ty)+d(y,Tx)} \varphi(t)dt \\ & + \beta \int_0^{\max\{d(x,Tx),d(y,Ty)\}} \varphi(t)dt \\ & - \int_0^{\psi\{d(x,Ty),d(y,Tx),d(x,Tx),d(y,Ty)\}} \varphi(t)dt \\ & + \delta \int_0^{\max\{d(y,Tx),d(y,Ty)\}} \varphi(t)dt \qquad (2.1)\end{aligned}$$

Then T has a unique fixed point.

Where $\alpha, \beta \in [0,1)$ with $2\alpha + \beta + \delta \leq 1$ and $\varphi : R^+ \to R^+$ is a Lebesgue – integrable mapping which is summable,non negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$ and $\Psi: [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that Ψ (x,y) = 0 if and only if x = y= 0 .

Proof : Let $x_0 \in X$ and for all n≥ 1 , $x_{n+1} = Tx_n$.

If $x_{n+1} = x_n = Tx_n$. Then x_n is a fixed point of T.

So we assume, $x_{n+1} \neq x_n$.

Putting $x = x_{n-1}$ and $y = x_n$ in (2.1) we have for all n = 0,1,2, …….

$$\begin{aligned}\int_0^{c(x_n,x_{n+1})} \varphi(t)dt =\ & \int_0^{d(Tx_{n-1},Tx_n)} \varphi(t)dt \\ \leq\ & \alpha \int_0^{d(x_{n-1},\ Tx_n)+d(x_n,Tx_{n-1})} \varphi(t)dt \\ & + \beta \int_0^{\max\{d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n)\}} \varphi(t)dt \\ & - \int_0^{\psi\{d(x_{n-1},Tx_n),d(x_n Tx_{n-1}),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n)\}} \varphi(t)dt \\ & + \delta \int_0^{\max\{d(x_n,Tx_{n-1}),d(x_n,Tx_n)\}} \varphi(t)dt \\ =\ & \alpha \int_0^{c(x_{n-1},\ x_{n+1})+d(x_n,x_n)} \varphi(t)dt\end{aligned}$$

$$+ \beta \int_0^{\max\{ d(x_{n-1},x_n), d(x_n,x_{n+1})\}} \varphi(t)dt$$
$$+\delta \int_0^{\max\{d(x_n,x_n), d(x_n,x_{n+1})\}} \varphi(t)dt$$

$$- \int_0^{\psi\{ d(x_{n-1},\ x_{n+1}), d(x_n x_n), d(x_{n-1},x_n), d(x_n,x_{n+1})\}} \varphi(t)dt$$

Since T is Weakly C – contraction, this gives that

$\psi\{ d(x_{n-1},\ x_{n+1}), 0, d(x_{n-1},x_n), d(x_n,x_{n+1})\} = 0$ and

$$\int_0^{c(x_n,x_{n+1})} \varphi(t)dt \le \alpha \int_0^{d(x_{n-1},\ x_{n+1})} \varphi(t)dt + \beta \int_0^{\max\{ d(x_{n-1},x_n), d(x_n,x_{n+1})\}} \varphi(t)dt +\delta \int_0^{\max\{d(x_n,x_n), d(x_n,x_{n+1})\}} \varphi(t)dt \tag{2.2}$$

Now here arise two cases

Case I: - If we choose

$\max\{ d(x_{n-1},x_n), d(x_n,x_{n+1}) \} = d(x_{n-1},x_n)$

Then (2.2) can be written as

$$\int_0^{c(x_n x_{n+1})} \varphi(t)dt \le \alpha \int_0^{d(x_{n-1},\ x_n)} \varphi(t)dt \ +\alpha \int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt +\beta \int_0^{d(x_{n-1},x_n)} \varphi(t)dt +\delta \int_0^{d(x_n,x_{n+1})\}} \varphi(t)dt$$

$$(1-\alpha-\delta) \int_0^{c(x_n,x_{n+1})} \varphi(t)dt = (\alpha+\beta) \int_0^{d(x_{n-1},x_n)} \varphi(t)dt$$

$$\int_0^{c(x_n,x_{n+1})} \varphi(t)dt = \frac{\alpha+\beta}{1-\alpha-\delta} \int_0^{d(x_{n-1},x_n)} \varphi(t)dt$$

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le K\int_0^{d(x_{n-1},x_n)} \varphi(t)dt \text{ where } k = \frac{\alpha+\beta}{1-\alpha-\delta} \le 1$$

***Case* 2** : : If we choose

$\max\{ d(x_{n-1},x_n), d(x_n,x_{n+1}) \} = d(x_n,x_{n+1})$

Then (2.2) can be written as

$$\int_0^{c(x_n,x_{n+1})} \varphi(t)dt \le \alpha \int_0^{d(x_{n-1},\ x_n)} \varphi(t)dt \ +\alpha \int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt +\beta \int_0^{d(x_n,x_{n+1})} \varphi(t)dt +\delta \int_0^{d(x_n,x_{n+1})\}} \varphi(t)dt$$

$$[1-(\alpha+\beta+\delta)]\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \alpha \int_0^{d(x_{n-1},\ x_n)} \varphi(t)dt$$

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \frac{\alpha}{1-(\alpha+\beta+\delta)} \int_0^{d(x_{n-1},\ x_n)} \varphi(t)dt$$

$$\int_0^{c(x_n,x_{n+1})} \varphi(t)dt \le k \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \quad ,\text{where } k = \frac{\alpha}{1-(\alpha+\beta+\delta)} \le 1 \tag{2.3}$$

From above both cases:

$$\int_0^{c(x_n,x_{n+1})} \varphi(t)dt \le k^2 \int_0^{d(x_{n-2},x_{n-1})} \varphi(t)dt$$
$$\le k^3 \int_0^{d(x_{n-3},x_{n-2})} \varphi(t)dt$$
$$\le \text{------}$$
$$\le k^n \int_0^{d(x_0,x_1)} \varphi(t)dt$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0, \text{ as } k \in [0,1) \tag{2.4}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. Suppose it is not. Then there exists an $\varepsilon > 0$ and sub sequence $\{y_{m(p)}\}$and $\{y_{n(p)}\}$such that

m(p) $<$ n(p) $<$ m(p+1) with

$$d(x_{n(p)},x_{m(p)}) \ge \varepsilon,\ d(x_{n(p)-1},x_{m(p)}) < \varepsilon \tag{2.5}$$

Now

$$d(x_{m(p)-1},x_{n(p)-1}) \le d(x_{m(p)-1},x_{m(p)}) + d(x_{m(p)},x_{n(p)-1})$$
$$< d(x_{m(p)-1},x_{m(p)}) + \varepsilon \tag{2.6}$$

From (2.4), (2.6) we get

$$\lim_{p\to\infty} \int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)dt \le \int_0^{\varepsilon} \varphi(t)dt \qquad (2.7)$$

Using (2.3), (2.5), and (2.7) we get,

$$\int_0^{\varepsilon} \varphi(t)dt \le \int_0^{d(x_{n(p)},x_{m(p)})} \varphi(t)dt$$

$$\le k\int_0^{d(x_{n(p)-1},x_{m(p)-1})} \varphi(t)dt$$

$$\le k\int_0^{\varepsilon} \varphi(t)dt$$

Which is contradiction, since k $\in$ (0, 1). therefore $\{x_n\}$ is a Cauchy sequence Since (X,d) is complete metric space , therefore have call the limit z .

From (2.1), we get

$$\int_0^{c(Tz,x_{n+1})} \varphi(t)dt = \int_0^{d(Tz,Tx_n)} \varphi(t)dt$$

$$\le \alpha \int_0^{d(z,Tx_n)+d(x_n,Tz)} \varphi(t)dt$$

$$+ \beta \int_0^{\max\{d(z,Tz),d(x_n,Tx_n)\}} \varphi(t)dt$$

$$- \int_0^{\psi\{d(z,Tx_n),d(x_n,Tz),d(z,Tz),d(x_n,Tx_n)\}} \varphi(t)dt$$

$$+ \delta \int_0^{\max\{d(x_n,Tz),d(x_n,Tx_n)\}} \varphi(t)dt$$

Taking limit as n $\to \infty$, we get

$$\int_0^{c(Tz,z)} \varphi(t)dt \le \alpha \int_0^{d(z,Tz)} \varphi(t)dt + \beta \int_0^{d(z,Tz)} \varphi(t)dt$$

$$= (\alpha + \beta) \int_0^{d(z,Tz)} \varphi(t)dt$$

Which is Contradiction

Therefore $Tz = z$

That is z is a fixed point of T in X.

Uniqueness : Let w is another fixed point of T in X such that z $\neq$ w , then we have

From (2.1), we get

$$\int_0^{c(z,w)} \varphi(t)dt = \int_0^{d(Tz,Tw)} \varphi(t)dt$$

$$\le \alpha \int_0^{d(z,Tw)+d(w,Tz)} \varphi(t)dt + \beta \int_0^{\max\{d(z,Tz),d(w,Tw)\}} \varphi(t)dt$$

$$- \int_0^{\psi\{d(z,Tw),d(w,Tz),d(z,Tz),d(w,Tw)\}} \varphi(t)dt + \delta \int_0^{\max\{d(w,Tz),d(w,Tw)\}} \varphi(t)dt$$

$$\int_0^{c(z,w)} \varphi(t)dt \le 2\alpha \int_0^{c(z,w)} \varphi(t)dt$$

Which is contradiction

So z = w that is , z is unique fixed point of T in X .

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